A Discussion of the Systematic Uncertainties in UTC-UTC(k)

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Abstract -- **The problem of computing the systematic uncertainties in UTC-UTC(k) was the subject of two contradictory papers that were published almost back-to-back last year [1,2]. In this paper the difference between the philosophies behind those papers is presented, and an example is shown to demonstrate that the approach of [2] can only be used to compute the uncertainties if the full history of UTC's link structure and calibration uncertainties is taken into account, stretching back to the initialization of UTC over fifty years ago.**

Keywords- UTC, uncertainties, timescale

I. INTRODUCTION

UTC (Coordinated Universal Time) is currently computed using formulas published in 2006 [3], which are consistent with [1], although it is anticipated that the BIPM will revise its approach shortly to be consistent with [2].

A common theme in all these papers is that UTC, EAL, and TAI are equivalent for the purpose of computing the uncertainties, UTC. This is because the differences between are known quantities with no uncertainty. Another similarity is that all the clocks in any lab k have the same uncertainty, because the uncertainties in UTC minus UTC(k), the real-time realization of UTC by lab k, are assumed to be due to the uncertainties of their time-transfer links. This would apply to any clock brought to lab k and measured locally against lab k's clocks.

All papers agree about the basic equation for EAL:

$$EAL = \sum_{n=1}^{N} w_n (h_n + h'_n) \qquad (1)$$

, where w_n is normalized weight of clock n, h_n is clock reading at the epoch in question and h'_n is the prediction, based on clock data, and the summation is over all contributing clocks[4]

Following equation (1) the uncertainty of UTC-UTC(k) is the same as the uncertainty of $EAL - h_k$

for any clock reading made locally against any clock in lab k. This can be written

$$EAL - h_k = \sum_{n=1}^{N} [w_n (h_n - h_k) + h'_k] \qquad (2)$$

Equivalently,

$$EAL - h_k = \sum_{n=1}^{N} w_n (h_n - h_k) + \sum_{n=1}^{N} w_n h'_n \quad (3)$$

In equations (2) and (3) the only measurements are the $h_n - h_k$ within the summations. The initial conditions and subsequent history of the timescale are embodied in the predictions, which are mathematical constructs based on past measurements. The principle, but not the only, difference between the contradictory papers is on how they treat the predictions. Even here, the papers agree with that clock predictions are not relevant for the purpose of computing statistical uncertainties. It may be however, that the BIPM is preparing to abandon publishing statistical uncertainties in favor of a frequency uncertainty, but that is beyond the scope of this paper.

The authors of both papers agree that, since the predictions are based on past data, they are contaminated by past biases in the time-transfer. Systematic link uncertainties are assumed to be equal to the uncertainties of the biases, at least since the most recent calibration of each link. According to paper [1], once the clock predictions are computed "the damage is done", and therefore the uncertainty of UTC-UTC(k) is given by the RSS of all the terms inside the summation of equation (3).

The philosophy behind paper [2] is to assume that the bias contaminating the clock predictions of each lab cancel the bias in the measurements involving the clocks. This is best understood if one considers a network in which each lab uses a GNSS receiver system for the time-transfer links, and it is the same GNSS. (Hereafter only GNSS links will be considered.) Assume also that the calibration of each lab's time-transfer system is already incorporated into the measurements, but a residual unknown calibration error b_n, remains. Statistically speaking, these residual unknown biases would have zero mean and an RMS/StD equal to the calibration uncertainty. Assume also that when a new lab k joins UTC, its clock predictions are based upon unweighted observations for many preceding epochs. They would have an unknown contamination b_k and that contamination would of course be incorporated into the prediction for lab k's clocks . Once lab k becomes a weighted contributor to UTC, all measurements h_n -h_k would have the same contamination (b_n -b_k), and this is neatly cancelled by the contamination of the clock predictions. Therefore the bias of lab k's time-transfer system would not affect the other labs.

The authors of both [1] and [2] agree with the above. The difference is that the author of [1] computes the uncertainty on the basis of the uncertainty of the epoch's measurement given the predictions, while the authors of [2] believe that the biases in the clock predictions always cancel the unmodeled biases in the time-transfer equipment. This leads to what is termed the "plausible solution" below.

I would hope that all authors would agree with the example given in [1], in which user making local measurements with lab k's clocks (or lab k itself) sets up independent bias-free links to all the labs (such as by high-quality optical fibers or "perfectly" calibrated GNSS receiver systems) and then uses those links for the time-transfer would find a deviation consistent with paper [1]'s formula for the uncertainty, and not just a deviation consistent with the uncertainty of lab k's GNSS system. Also simulations validating the formulas of [1] were reported in [5] for the case in which the clock predictions for the first two epochs were bootstrapped from the data of the first two epochs. Paper [5] criticized some of the mathematics in [2], but neglected to point out that [2] starts out assuming what it was trying to derive. That is, its fifth and sevenths equations assume that for a GNSS-linked lab k, the bias in its GNSS equipment equaled the bias of its UTC-UTC(k). However, the purpose of this paper is not to belabor this point.

The purpose of this paper is to follow the reasoning of [2], and to present two examples for an all-GNSS non-redundant network that show the reasoning of [2] can not consistently be employed to compute the systematic uncertainties of UTC-UTC(k) if it only takes into consideration the uncertainties of the current links calibration and the clock weights. Rather, the full history of UTC must be considered, including all the link uncertainties and associated weights used since UTC was initiated.

II. A PLAUSIBLE SOLUTION FOR AN ALL-GNSS NETWORK

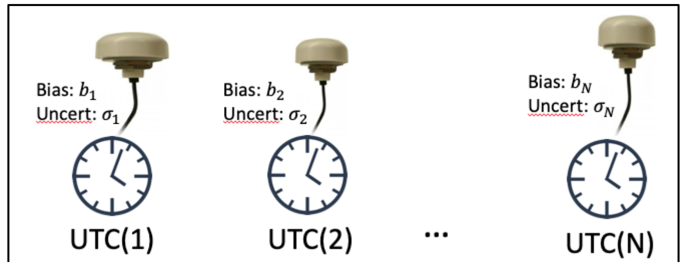

Figure 1. A network of GNSS-link laboratories

In the network of figure 1, one might assume the bias and uncertainty of lab k's GNSS system (b_k and $\sigma_{k,GNSS}$) equal the bias and uncertainty of UTC-UTC(k). This equivalence of these biases is what is used in [2] to derive its subsequent formulas.

However, the simple picture breaks down if there are two GNSS involved, even if still non-redundant.

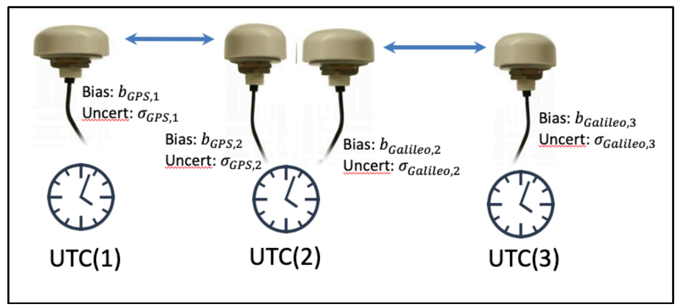

Figure 2 A three-lab topology, which involves two different GNSS systems.

In this case, the formula can not be applied because it is not obvious which GNSS's bias (GPS or Galileo) should be assigned to lab 2. It depends on the history, as is shown in the next section.

III. INITIALIZING UTC FROM ONE LAB

If initially only one lab is contributing to UTC (Figure 3), then there is no time-transfer contamination of the clock predictions. If that lab has only one clock, the prediction is zero. If there are more clocks, the predictions depend on the locally measured clock differences. For the purposes of this paper, those can be ignored with no loss of generality, so again the predictions are zero.

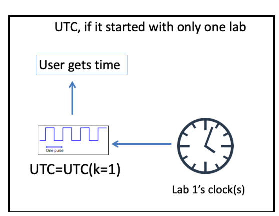

Figure 3. A one-lab network for UTC.

If a second lab wishes to join UTC (Figure 4), its clocks will first be compared to the original lab's clocks so as to generate clock predictions.

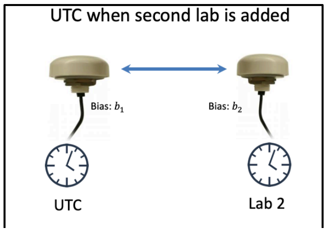

Figure 4 A second lab is beginning to participate in UTC.

Ignoring the clock variations and considering only the contamination of the data and of the predictions due to the link biases, the data over evaluation period are

$$UTC - h_2 = h_{1,0} - h_{2,0} + b_1 - b_2 \qquad (4)$$

Where $h_{n,0}$ is the actual clock reading that would be revealed if there were no noise or biases in the time-transfer data.

As a result of the evaluation data, the contamination of the prediction for lab 2 would be:

$$h'_2 = b_1 - b_2 \qquad (5)$$

And the prediction for lab 1 would not be affected so its contamination would remain 0:

$$h'_1 = 0 \qquad (6)$$

Although the clock variations are ignored, one could use the notation $h'_{n,o}$ to describe the clock prediction as it would have been in the absence of systematic errors (unknown biases). These terms would have no uncertainty and drop out along the way.

Once the data from lab 2 are weighted, the equation for EAL becomes

$$EAL = w_1(h_1 + h'_1) + w_2(h_2 + h'_2) \qquad (7)$$

Ignoring the clock variations and considering only the contamination due to the biases,

$$EAL = w_1(h_1 + 0) + (1 - w_1)(h_2 + b_1 - b_2) \qquad (8)$$

The uncertainties of UTC-UTC(k), for k=1 to 2 equal the uncertainties of

$$EAL - h_1 = w_1(h_1 - h_1) + (1 - w_1)(h_2 - h_1 + b_1 - b_2) \qquad (9)$$

$$EAL - h_2 = w_1(h_1 - h_2) + (1 - w_1)(h_2 - h_2 + b_1 - b_2) \qquad (10)$$

If one considers only the bias in the $h_1 - h_2$ measurement (which is $b_1 - b_2$), equations (9) and since there is no bias in the local measurements of $h_n - h_n$, equations (9) and (10) become:

$$EAL - h_1 = (w_1 - 1)0 + (1 - w_1)(0) = 0 \qquad (11)$$

$$EAL - h_2 = w_1(b_1 - b_2) + (1 - w_1)(b_1 - b_2) = b_1 - b_2 \qquad (12)$$

Again, only the bias-contamination terms in $EAL - h_n$ are retained. Since the unknown biases are zero mean and have a normal distribution with variance $\sigma^2_{n,GNSS}$, it is trivial to derive the uncertainties in UTC-UTC(k):

$$\sigma^2_{UTC-UTC(1)} = 0 \qquad (13)$$

$$\sigma^2_{UTC-UTC(2)} = \sigma^2_{1,GNSS} + \sigma^2_{2,GNSS} \qquad (14)$$

These results are inconsistent with all papers. Also, if lab 2 had been the first lab to define UTC, then the uncertainties would be reversed. This shows that the history is relevant and that the formulas of (2), even if valid, must be incomplete.

IV. DISCUSSION

Perhaps the discrepancy between the philosophy of paper [2] and that of the previous works can be resolved by nomenclature. The systematic uncertainties computed using the algorithms of the earlier papers can be called the "user's systematic uncertainties". These would be as are now shown, although minor adjustments could be made following suggestions in [1] and [5]. If one is willing to ignore the fact that the unknown biases become part of the scaffold once they have corrupted the clock predictions, those computed using the philosophy of paper [2] and the algorithm sketched herein can be termed "prediction-subtracted uncertainties". These would be similar, but not equal to, the values in Section 5 of the Circular T as well as [2].

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